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TWO BOUNDARY VALUE PROBLEMS FOR A STRONGLY ANISOTROPIC INHOMOGENEOUS ELASTIC RING*

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The second boundary value problem (displacements are given on the boundary) and the improper mixed problem for a cylindrically orthotropic ring are studied. It is assumed that the coefficients of elasticity are continuously differentiable functions of the coordinates and depend on a small parameter in a specific manner. The form of the dependence of the coefficients on the small parameter is selected in such a way that in the case of constant coefficients it describes bonding of the ring by two families of very rigid fibers located along the radius vectors and concentric circles, where the stiffness of the fiber families is of identical order. Consequently, the coefficients of elasticity are represented in the form of products of constants which will later be called provisionally the "stiffnesses", and functions of the coordinates. It is assumed that the stiffnesses in the radial and circumferential directions are equal and exceed and shear stiffness considerably. The asymptotic form of the solution of the boundary value problems under consideration is constructed when the ratio between the shear stiffness and the stiffness in the radial direction is used as the small parameter. In the case of the second boundary value problem the limit boundary value problem is described by a hyperbolic system of equations and is not solvable uniquely, since one of the families of characteristics is parallel to the boundary. When constructing the asymptotic form the necessity arises to average the coefficients of elasticity with respect to the circumferential coordinate. In this respect, there is an analogy with the results obtained in /1/ where the boundary value problem was studied for a second-order elliptic equation.

1. We take the generalized Hooke's law in the form

$$\sigma_r = c_{11}d_1e_r + c_{12}d_2e_\theta,\quad \tau_{r\theta} = c_{66}d_4e_{r\theta},\quad \sigma_\theta = c_{12}d_2e_r + c_{11}d_3e_\theta$$

$$e_r = \frac{\partial u}{\partial r},\quad e_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r},\quad e_{r\theta} = \frac{1}{r}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

where c_{11}, c_{12}, c_{66} are constants, d_k $(k = 1, 2, 3, 4)$ are continuous differentiable functions of the coordinates, and u, v are the radial and circumferential displacement. From the fact that the strain potential energy is positive it follows that c_{ij}, d_i should satisfy the constraints

$$c_{11}^2d_1d_3 - c_{12}^2d_2^2 > 0,\quad c_{11}d_1 > 0,\quad c_{66}d_4 > 0$$

Let $c_{11} > 0$, $c_{66} > 0$, $c_{11} \gg c_{66}$. We introduce the small parameter $\varepsilon^2 = c_{66}c_{11}^{-1}$ and the dimensionless stresses by setting

$$\bar\sigma_r = \sigma_r c_{11}^{-1},\quad \bar\sigma_\theta = \sigma_\theta c_{11}^{-1},\quad \bar\tau_{r\theta} = \tau_{r\theta} c_{11}^{-1}$$

We subsequently keep the previous notation for the dimensionless stresses. Then the generalized Hooke's law can be written as follows:

$$\sigma_r = d_1e_r + b\varepsilon^2d_2e_\theta,\quad \tau_{r\theta} = \varepsilon^2d_4e_{r\theta} \tag{1.1}$$

$$\sigma_\theta = b\varepsilon^2d_2e_r + d_3e_\theta,\quad b = c_{12}c_{11}^{-1}$$

Let Q be a circular ring, $Q = \{(r, \theta); 0 < c \leqslant r \leqslant a\}$. We introduce the dimensionless coordinate $x = \ln(r/a)$ and we set $x_0 = \ln(c/a)$. We write the system of equilibrium equations when there are no volume forces in the form

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$$d_1 \frac{\partial^2 u}{\partial x^2} - d_3\left(u + \frac{\partial v}{\partial \theta}\right) + \varepsilon^2 \left[\frac{\partial}{\partial x}\left(d_2\left(u + \frac{\partial v}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(d_4\left(\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} - v\right)\right)\right] = 0 \qquad (1.2)$$

$$\frac{\partial}{\partial \theta}\left[d_3\left(u + \frac{\partial v}{\partial \theta}\right)\right] + \varepsilon^2 \left[b \frac{\partial}{\partial \theta} d_2 \frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(d_4 \frac{\partial u}{\partial \theta} + d_4 \frac{\partial v}{\partial x}\right) - \left(\frac{\partial d_4}{\partial x} + d_4\right)\left(v - \frac{\partial u}{\partial \theta}\right)\right] = 0$$

We pose the following boundary value problems for (1.2) Problem A_ε:

$$u(0, \theta) = p_1(\theta), \quad u(x_0, \theta) = p_2(\theta) \qquad (1.3)$$
$$v(x_0, \theta) = p_4(\theta), \quad v(0, \theta) = p_3(\theta)$$

Problem B_ε: the first three boundary conditions of problem A_ε are preserved and the following condition is substituted for the fourth

$$\left[\varepsilon^2 d_4\left(\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} - v\right) + \rho\left(d_1 \frac{\partial u}{\partial x} + b\varepsilon^2 d_2\left(u + \frac{\partial v}{\partial \theta}\right)\right)\right]_{x=0} = 0 \qquad (1.4)$$

corresponding to the presence of a dry friction force on the boundary of the body, $x = 0$, where ρ is the coefficient of friction /2/. We require that the displacements and the functions $p_k(\theta)$ $(k = 1, 2, 3, 4)$ be continuously differentiable functions of the polar angle.

2. We will construct the astmptotic form of the problem A_ε for small ε. We will seek the approximate solution of (1.2) in the form

$$u^\circ(x, \theta) = \sum_{n=0}^{N} \varepsilon^n u_n(x, \theta), \quad v^\circ(x, \theta) = \sum_{n=0}^{N} \varepsilon^n v_n(x, \theta) \qquad (2.1)$$

where u_n and v_n are perodic functions of θ. Substituting (2.1) into (1.2) we obtain a recurrent coupled system of equations

$$d_1 \frac{\partial^2 u_n}{\partial x^2} - d_3\left(u_n + \frac{\partial v_n}{\partial \theta}\right) = P_n(u_{n-2}, v_{n-2}) \qquad (2.2)$$
$$\frac{\partial}{\partial \theta}\left[d_3\left(u_n + \frac{\partial v_n}{\partial \theta}\right)\right] = Q_n(u_{n-2}, v_{n-2})$$
$$P_0 = P_1 = Q_0 = Q_1 = 0$$

(P_n, Q_n are differential operators for the powers ε^2 in (1.2) in the first and second equations, respectively).

Let us examine (2.2) for $n = 0$. We require that the function $u_0(x, \theta)$ satisfy the first two boundary conditions (1.3). Integrating the second equation in (2.2), we obtain

$$v_0(x, \theta) = \frac{\langle u_0(x, \theta)\rangle}{\lambda(x)} \int_0^\theta \frac{ds}{d_3(x, s)} - \int_0^\theta u_0(x, s)\, ds + g_0(x), \quad \lambda(x) = \langle d_3^{-1}(x, \theta)\rangle$$

where $\langle m\rangle$ denotes the mean of the function $m(x, \theta)$ over the period, and $g_0(x)$ is arbitrary and not determined by using the first two boundary conditions in (1.3).

The first equation of system (2.2) acquires the form

$$d_1 \frac{\partial^2 u_0(x, \theta)}{\partial x^2} - \frac{\langle u_0(x, \theta)\rangle}{\lambda(x)} = 0 \qquad (2.3)$$

Representing $u_0(x, \theta)$ in the form of the trigonometric series

$$u_0(x, \theta) = \langle u_0(x, \theta)\rangle + \sum_{n=1}^{\infty} u_{n1}(x) \cos n\theta + u_{n2}(x) \sin n\theta$$

it can be shown that (2.3) has a unique solution for the first two boundary conditions (1.3). Consequently, the function $u_0(x, \theta)$ is defined uniquely, while $v_0(x, \theta)$ is defined apart from the arbitrary function $g_0(x)$. Continuing the iteration procedure further, it can be shown that even for $n \geqslant 1$ the functions $v_n(x, \theta)$ are defined just to the accuracy of an arbitrary function $g_n(x)$.

The equation to determine the functions $g_n(x)$ can be obtained from the second equation of (2.2). In fact, in order for it to have a periodic solution, it is necessary and sufficient that the mean of the right side with respect to the period should be equal to zero, and hence, the following equation should hold:

$$\frac{\partial}{\partial x}\left[\langle d_4(x, \theta)\rangle \frac{\partial v_n}{\partial x}\right] - \langle m(x, \theta)\rangle v_n = 0, \quad m(x, \theta) = \frac{\partial d_4}{\partial x} + d_4$$

But $v_n = q_{n,0}(x, \theta) + g_n(x)$, where $q_{n,0}(x, \theta)$ is a known periodic function. We hence obtain the following equation for the function $g_n(x)$:

$$\frac{d}{dx}\left[\langle d_4(x, \theta)\rangle \frac{dg_n}{dx}\right] - \langle m(x, \theta)\rangle g_n = 0 \tag{2.4}$$

The solution of (2.4) depends on two arbitrary constants $g_n(0)$ and $g_n(x_0)$ to determine which we use the boundary layer functions.

We construct the boundary layer functions near $x = 0$ (they are constructed analogously near $x = x_0$). We introduce the stretching coordinate $\eta = x/\varepsilon$ near $x = 0$. We expand the coefficients of (1.2) in a Taylor series in powers of ε and we seek the approximate solution of the system of equations obtained in the following form:

$$u^1(\eta, \theta) = \varepsilon^2 \sum_{n=0}^{N-2} \varepsilon^n u_{n,0}(\eta, \theta),\ v^1(\eta, \theta) = \sum_{n=0}^{N} \varepsilon^n v_{n,0}(\eta, \theta) \tag{2.5}$$

We obtain a recurrent coupled system of equations

$$\begin{aligned} & d_1(0, \theta)\frac{\partial^2 u_{n,0}}{\partial\eta^2} - d_3(0, \theta)\frac{\partial v_{n,0}}{\partial\theta} = f_{n,0}(u_{k,0};\ v_{k,0}) \\ & \frac{\partial}{\partial\theta}\left[d_3(0, \theta)\frac{\partial v_{n,0}}{\partial\theta}\right] + \frac{\partial}{\partial\eta}\left[d_4(0, \theta)\frac{\partial v_{n,0}}{\partial\eta}\right] = g_{n,0}(u_{k,0};\ v_{k,0}) \\ & f_{0,0} = g_{0,0} = 0,\ k < n \end{aligned} \tag{2.6}$$

to determine the functions $u_{n,0}, v_{n,0}$, where $f_{n,0}, g_{n,0}$ are known differential operators. We require that the functions $v_{n,0}(0, \theta)$ be periodic in θ and decrease exponentially as $\eta \to +\infty$. The function $v_{0,0}(\eta, \theta)$ should satisfy the boundary condition

$$v_{0,0}(0, \theta) = p_3(\theta) - q_{0,0}(0, \theta) - g_0(0)$$

According to Lemma 5 in /1/, the solution of (2.6) that is bounded as $\eta \to +\infty$ admits of the estimate

$$| v_{0,0}(\eta, \theta) - \langle p_3(\theta) - q_{0,0}(0, \theta) - g_0(0)\rangle | < c_1 e^{-c_2\eta},\ c_1 > 0, \qquad c_2 > 0$$

and it is necessary to require that the integral in the preceding inequality be zero for the decrease in $v_{0,0}(\eta, \theta)$ to be exponential. Hence

$$g_0(0) = \langle p_3(\theta) - q_{0,0}(0, \theta)\rangle$$

By constructing the boundary layer functions near $x = x_0$ we similarly obtain $g_0(x_0) = \langle p_4(\theta) - q_{0,0}(x_0, \theta)\rangle$. Knowing $g_0(0)$, $g_0(x_0)$, we determine $g_0(x)$ and $v_0(x, \theta)$ from (2.4). Performing the iteration procedure of constructing the boundary layer functions further, we determine $g_n(0)$ and $g_n(x_0)$, and consequently, $v_n(x, \theta)$.

Finally, the asymptotic form of problem A_ε has the form

$$u(x, \theta) = \sum_{n=0}^{N} \varepsilon^n u_n(x, \theta) + \varepsilon^2 \sum_{n=0}^{N-2} \varepsilon^n [u_{n,0}(\eta, \theta) + u_{n,1}(\eta, \theta)] + \varepsilon^{N+1} R_N^{(1)}(x, \theta)$$

$$v(x, \theta) = \sum_{n=0}^{N} \varepsilon^n [v_n(x, \theta) + v_{n,0}(\eta, \theta) + v_{n,1}(\eta_1, \theta)] + \varepsilon^{N+1} R_N^{(2)}(x, \theta)$$

where $u_{n,1}(\eta_1, \theta)$, $v_{n,1}(\eta_1, \theta)$ are the boundary layer functions near $x = x_0$, $\eta_1 = (x_0 - x)/\varepsilon$, $\varepsilon^{N+1} R_N^{(k)}$ $(k = 1, 2)$ are the remainder terms.

3. Let us construct the asymptotic form of problem B_ε for small ε . Unlike problem A_ε the solution of (1.2) must be sought in the form

$$u^\circ(x, \theta) = \sum_{n=0}^{N} \varepsilon^n u_n(x, \theta),\ \ v^\circ(x, \theta) = \varepsilon^{-2} \sum_{n=0}^{N} \varepsilon^n v_n(x, \theta) \tag{3.1}$$

Substituting (3.1) into (1.2) we obtain the recurrent coupled system of equations

$$d_3 \frac{\partial v_n}{\partial\theta} = 0,\ \frac{\partial}{\partial\theta}\left(d_3 \frac{\partial v_n}{\partial\theta}\right) = 0,\quad n = 0, 1 \tag{3.2}$$

$$\begin{aligned} & -d_3 \frac{\partial v_n}{\partial\theta} + L(u_{n-2}, v_{n-2}) = 0 \\ & \frac{\partial}{\partial\theta}\left(d_3 \frac{\partial v_n}{\partial\theta}\right) + M(u_{n-2}, v_{n-2}) = 0,\quad n = 2, 3 \end{aligned} \tag{3.3}$$

$$-d_3\frac{\partial v_n}{\partial\theta}+L(u_{n-2},v_{n-2})+L_1(u_{n-4},v_{n-4})=0 \tag{3.4}$$

$$\frac{\partial}{\partial\theta}\left(d_3\frac{\partial v_n}{\partial\theta}\right)+M(u_{n-2},v_{n-2})+M_1(u_{n-4},v_{n-4})=0,\quad n\geqslant 4$$

where the differential operators $L(u,v)$, $M(u,v)$, $L_1(u,v)$, $M_1(u,v)$ are given by the formulas

$$L(u,v)=d_1\frac{\partial^2u}{\partial x^2}-d_3u+\frac{\partial}{\partial x}\left(d_2\frac{\partial v}{\partial\theta}\right)+\frac{\partial}{\partial\theta}\left[d_4\left(\frac{\partial v}{\partial x}-v\right)\right]$$

$$M(u,v)=\frac{\partial}{\partial\theta}(d_3u)+\frac{\partial}{\partial x}\left(d_4\frac{\partial v}{\partial x}\right)-mv$$

$$L_1(u,v)=\frac{\partial}{\partial x}(d_2u)+\frac{\partial}{\partial\theta}\left(d_4\frac{\partial u}{\partial\theta}\right)$$

$$M_1(u,v)=m\frac{\partial u}{\partial\theta}+b\frac{\partial}{\partial\theta}\left(d_2\frac{\partial u}{\partial x}\right)+b\frac{\partial}{\partial x}\left(d_4\frac{\partial u}{\partial\theta}\right)$$

We will examine the question of the existence of periodic solutions of (3.2)–(3.4). It obviously follows from (3.2) that $v_0=v_0(x)$, $v_1=v_1(x)$. Integrating the second equation of (3.3) with respect to θ for $n=2$ we obtain

$$d_3\frac{\partial v_2}{\partial\theta}+d_3u_0+\frac{\partial}{\partial x}\left[\int_0^\theta d_4(x,s)\frac{\partial v_0(x,s)}{\partial x}ds\right]-v_0(x)\int_0^\theta m(x,s)\,ds+g_0(x)=0 \tag{3.5}$$

It follows from the periodicity of u_0 that v_0 satisfies the equation

$$\frac{\partial}{\partial x}\left[\langle d_4\rangle\frac{\partial v_0}{\partial x}\right]-\langle m\rangle v_0=0$$

Combining (3.5) with the first equation of (3.3) for $n=2$, we obtain

$$d_1\frac{\partial^2u_0}{\partial x^2}+\frac{\partial d_4}{\partial\theta}\left(\frac{\partial v_0}{\partial x}-v_0\right)+\frac{\partial}{\partial x}\left[\int_0^\theta d_4(x,s)\frac{\partial v_0}{\partial x}ds\right]-v_0(x)\int_0^\theta m(x,s)\,ds+g_0(x)=0$$

where the function $g_0(x)$ is determined from the periodicity condition for $v_2(x,\theta)$ in θ.

We multiply (3.5) by d_3^{-1} and integrate the result with respect to θ. Then

$$v_2(x,\theta)+\int_0^\theta u_0(x,s)\,ds+S(v_0,x)+\theta g_0(x)+g_1(x)=0$$

where $S(v_0,x)$ is a known function. It follows from the periodicty of $v_2(x,\theta)$ that $g_0(x)=-\langle u_0\rangle+G_0(v_0,x)$, ($G_0(v_0,x)$ is a known function), which yields an equation for $u_0(x,\theta)$

$$d_1\frac{\partial^2u_0}{\partial x^2}-\langle u_0(x,\theta)\rangle+G_0(v_0,x)+G_1(v_1,x)=0 \tag{3.6}$$

analogous to (2.3) and allowing a unique solution for given $u_0(0,\theta)$, $u_0(x_0,\theta)$. Here $v_2(x,\theta)=q_{20}(x,\theta)+g_2(x)$, where $q_{20}(x,\theta)$ is determined from the known functions u_0, v_0 and is periodic in θ, and $g_2(x)$ must be determined.

To do this, we substitute $v_2(x,\theta)=q_{20}(x,\theta)+g_2(x)$ into the second equation of (3.4) for $n=4$ and we use the condition of periodicity of v_4 in θ. We hence have an equation for $g_2(x)$

$$\frac{d}{dx}\left[\langle d_4\rangle\frac{dg_2}{\partial x}\right]-\langle m(x,\theta)\rangle g_2+F_0(x)=0$$

analogous to (2.4), and for u_2 an equation analogous to (3.6).

Continuing the iteration process further, we obtain that $u_n(x,\theta)$ is determined from the equation

$$d_1\frac{\partial^2u_n}{\partial x^2}-\langle u_n(x,\theta)\rangle=G_n(x)$$

and $v_n(x,\theta)=q_{n0}(x,\theta)+g_n(x)$, where q_{n0} are determined in terms of the functions u_k, v_k of the preceding iterations, where $g_n(x)$ satisfy the equation

$$\frac{d}{dx}\left[\langle d_4\rangle\frac{dg_n}{dx}\right]-\langle m\rangle g_n=F_n(x)$$

and $F_n(x)$ are also known. The boundary layer functions near $x=0, x_0$ must be known for a complete determination of $g_n(x)$.

The boundary layer functions are sought in the form (2.5). The procedure for constructing them is analogous to that described above with the sole difference that the functions $v_{n,0}(\eta,\theta)$ will satisfy boundary conditions of the Neumann type for $\eta = 0$, which enables $g_n'(0)$, $g_n'(x_0)$ to be determined from the condition for the damping of the boundary layer functions to be exponential /3/.

The asymptotic form of problem B_ε differs substantially from the asymptotic form of problem A_ε in that the series expansion in powers of ε for $v(x,\theta)$ must start with the power -2, and this is related, in turn, to the fact that the coefficient of friction is assumed to be non-zero. For $\rho = 0$ the series expansion starts with the zeroth power of ε.

The system of equations (2.2) to determine the functions u_n, v_n is hyperbolic with two double families of characteristics $x \equiv \text{const}$ and $\theta \equiv \text{const}$, which indeed results in the appearance of the average with respect to the angular coordinate in the asymptotic form because of the requirement for the displacement to be unique. We note that the "radial" part of the functions $v_n(x,\theta)$ is extracted automatically in problem B_ε.

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THE PROBLEM OF THE CONTACT BETWEEN A LINEAR ELASTIC BODY AND ELASTIC AND RIGID BODIES (A VARIATIONAL APPROACH)*

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The problem of the contact between a linear elastic body and a rigid body is formulated as a one-sided problem. The solution is determined from the variational inequality, equivalent to the problem of minimizing the energy functional in a set of allowable displacements. The regularity of the solution is established down to internal points of the contact boundary. A measure is constructed in the subsets of the contact boundary that enables the effect of a stamp on an elastic body to be characterized. The absolute continuity of this measure is proved at the internal point. The problem of the contact of two elastic bodies is examined in a similar formulation. The regularity of the solution is established and the nature of the effect of one body on the other is clarified.

1. Contact between an elastic and a rigid body.

Formulation of the problem. Let an elastic body in the natural state occupy a domain $\Omega \subset R^3$ with boundary Γ of class C^∞ represented in the form of the union of three parts: $\Gamma = \Gamma_\omega \cup \Gamma_\sigma \cup \Gamma_c$. The condition $\omega = 0$ is given on Γ_ω, where ω is the displacement vector. The vector force $\sigma_{ij}n_j = g_i$ is given on Γ_σ, where $n = (n_1, n_2, n_3)$ is the external normal to the boundary, σ_{ij} is the stress tensor, g_i are given surface forces, $i, j =$ 1,2,3, and summation here and below is over repeated subscripts. It is assumed that the points Γ_c of the elastic body can interact with the rigid body for which the equation of the surface has the form $\Phi(x) = 0$, where the inequality $\Phi(x) \leqslant 0$ is satisfied for points of the rigid body. In the linear approximation the condition on the displacement vector has the form /1/

$$\omega(x)\nabla\Phi(x) \geqslant -\Phi(x), \quad x \in \Gamma_c \tag{1.1}$$

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